II. *On the Stability of Loose Earth.* *By* W. J. Macquorn Rankine, *F.R.S.*

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§ 1. *General Principle.*

The subject of this paper is,—the mathematical theory of that kind of stability, which, in a mass composed of separate grains, arises wholly from the mutual friction of those grains, and not from any adhesion amongst them.

Previous researches on this subject are based (so far as I am acquainted with them) on some mathematical artifice or assumption, such as Coulomb's "wedge of least resistance." Researches so based, although leading to true solutions of many special problems, are both limited in the application of their results, and unsatisfactory in a scientific point of view. I propose, therefore, to investigate the mathematical theory of the frictional stability of a granular mass, without the aid of any artifice or assumption, and from the following sole

Principle.

The resistance to displacement by sliding along a given plane in a loose granular mass, is equal to the normal pressure exerted between the parts of the mass on either side of that plane, multiplied by a specific constant.

The specific constant is the *coefficient of friction* of the mass, and is regarded as the tangent of an angle called the *angle of repose.* Let P denote the normal pressure per unit of area of the plane in question; F the resistance to sliding (per unit of area also); φ the angle of repose; then the symbolical expression of the above principle is as follows:—

$$\frac{F}{P}=\tan\varphi. \qquad (1.)$$

This principle forms the basis of every investigation of the stability of earth. The peculiarity of the present investigation consists in its deducing the laws of that stability from the above principle alone, without the aid of any other special principle. It will in some instances be necessary to refer to Mr. Moseley's "Principle of the Least Resistance;" but this must be regarded not a special principle, but as a general principle of statics.

§ 2. *Corollary as to Limit of Obliquity of Pressure.*

It is necessary to the stability of a granular mass, that the direction of the pressure between the portions into which it is divided by any plane should not at any point make with the normal to the plane an angle exceeding the angle of repose.

That is to say, symbolically, let R be the total pressure, per unit of area, at any point of the given plane, making with the normal to the plane the angle of obliquity θ; let P be the normal and Q the tangential component of R; so that

$$P=R\cos\theta; \qquad Q=R\sin\theta;$$

$$\frac{Q}{P}=\tan\theta;$$

then it is necessary to stability that

and consequently that

$$\left.\begin{array}{l} Q\leq F=P\tan\varphi, \\ \theta\leq\varphi. \end{array}\right\} \quad \text{(2.)}$$

§ 3. *Lemmata as to the Composition of the Stress at a point.*

It is well known that the stress at any point in a solid medium is capable of being resolved, with reference to any set of three rectangular axes, into six elements, viz. three normal pressures, P_x, P_y, P_z, on unity of area of the three coordinate planes, and three tangential pressures, Q_x, Q_y, Q_z, on unity of area of the three pairs of coordinate planes parallel to the three axes respectively. It is also known, that if we take these six elementary stresses for the coefficients of what, in Mr. CAYLEY'S nomenclature *, is called a *Ternary Quadric*, and in the nomenclature of a paper on Axes of Elasticity, a *Tasimetric Function* †,

$$U=P_xx^2+P_yy^2+P_zz^2+2Q_xyz+2Q_yzx+2Q_zxy, \quad \text{(3.)}$$

then if this quadric be transformed so as to be referred to new axes, the coefficients of the transformed quadric will be the elementary stresses referred to the new axes; and further, that there is a set of three rectangular axes, being the principal axes of the surface $U=1$, for which the tangential stresses vanish, and the normal stresses become maxima or minima, the quadric being reduced to

$$U=P_xx^2+P_yy^2+P_zz^2. \quad \text{(4.)}$$

The normal stresses for those principal axes of pressure are called the *principal pressures.*

Let P_x be the greatest and P_y the least of the three principal pressures at a given point O, and let On, making with Ox the angle $x\text{O}n=\psi$, be a line in the plane xy. Let R_n be the total pressure on unity of area of a plane normal to On, and let the direction of this pressure make with On the angle θ on the side of On towards x, so that the components of R_n are respectively, normal, $P_n=R_n\cos\theta$; tangential, $Q_n=R_n\sin\theta$.

Let the half-sum of the greatest and least principal pressures be denoted by

$$M=\frac{P_x+P_y}{2},$$

and their half-difference by

$$D=\frac{P_x-P_y}{2}.$$

* Philosophical Transactions for 1854–55, "On Quantics."

† Ibid. 1855.

Then the magnitude and direction of the pressure exerted at the plane normal to On, are given by the following equations:—

$$\left.\begin{aligned} R_n &= \sqrt{M^2+D^2+2MD\cos 2\psi} \\ \tan\theta &= \frac{D\sin 2\psi}{M+D\cos 2\psi}, \end{aligned}\right\} \quad \ldots \ldots \quad (5.)$$

or otherwise by the following:—

$$\left.\begin{aligned} P_n &= M+D\cos 2\psi \\ Q_n &= D\sin 2\psi. \end{aligned}\right\} \quad \ldots \ldots \quad (6.)$$

The *maximum* value Θ of the obliquity θ, and the corresponding position of the normal On, are given by the following equations:—

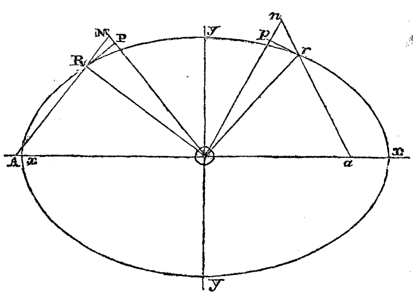
$$\left.\begin{aligned} \Psi &= \frac{\pi}{4}+\frac{1}{2}\sin^{-1}\frac{D}{M} \\ \Theta &= \sin^{-1}\cdot\frac{D}{M}, \end{aligned}\right\} \quad \ldots \ldots \quad (7.)$$

to which correspond the following pressures, total, normal, and tangential:—

$$\left.\begin{aligned} R(\Psi) &= \sqrt{M^2-D^2} \\ P(\Psi) &= M\left(1-\frac{D^2}{M^2}\right) \\ Q(\Psi) &= D\sqrt{1-\frac{D^2}{M^2}}. \end{aligned}\right\} \quad \ldots \ldots \quad (8.)$$

The following geometrical construction represents the theorems expressed symbolically by the equations (5.), (6.), (7.), (8.) (fig. 1). Let xOx, yOy be the axes of greatest and least pressure at the point O. Given, the half-sum M, and half-difference D, of those pressures, it is required to find:—

Fig. 1.



First, the direction and magnitude of the pressure at a plane normal to On, which line is in the plane xy, and makes with the axis of greatest pressure the given angle xOn=ψ. Make On=M. On Ox take the point a so that na=On. On na take nr=D. Then will Or=R_n represent the pressure required, and rOn=θ will be its angle of obliquity.

Also, let fall $rp \perp$ On; then Op=P_n, pr=Q_n.

Secondly, to find the plane for which the obliquity θ is greatest. Make xON=Ψ= half a right angle + half the angle whose sine is $\frac{D}{M}$. Then ON will be the normal to a plane having the required property. There is obviously a pair of such planes, whose normals make equal angles at either side of Ox.

The remainder of the construction is to be proceeded with as before, to find the total pressure $\overline{OR}$, and its normal and tangential components $\overline{OP}$, $\overline{PR}$. It is obvious that ORN is a right angle.

The locus of the points r is an ellipse, whose semiaxes $Ox = P_x$, $Oy = P_y$ represent the greatest and least pressures at the point O.

§ 4. *Additional Lemmata, as to the Transformation of Stress in the plane of greatest and least pressure.*

By the plane of greatest and least pressure at a given point O, is meant the plane containing the axes Ox and Oy of greatest and least pressure. Let there be taken any two new rectangular axes in that plane, Ox' and Oy', and let

$$< xOx' = < yOy' = \psi, \text{ so that } < xOy' = \frac{\pi}{2} + \psi.$$

Also let $P_{x'}$ and $P_{y'}$ be the normal pressures at planes perpendicular to the axes Ox', Oy', respectively, and Q' the tangential pressure on either of those planes. Then from equations (6.) it appears that

$$\left.\begin{array}{l} P_{x'} = M + D \cos 2\psi \\ P_{y'} = M - D \cos 2\psi \\ Q' = D \sin 2\psi. \end{array}\right\} \quad \ldots \ldots \quad (9.)$$

Consequently, if the elementary stresses $P_{x'}$, $P_{y'}$, Q', at any pair of planes at right angles to each other and to the plane of greatest and least pressure be given, the greatest and least pressures, and the positions of their axes, are given by the following equations:—

$$\left.\begin{array}{l} \frac{P_x + P_y}{2} = M = \frac{P_{x'} + P_{y'}}{2} \\ \frac{P_x - P_y}{2} = D = \sqrt{\left\{\frac{(P_{x'} - P_{y'})^2}{4} + Q'^2\right\}} \\ \tan 2\psi = \frac{2Q'}{P_{x'} - P_{y'}}. \end{array}\right\} \quad \ldots \ldots \quad (10.)$$

The equations given above solve a particular case only of the general problem, viz. the case in which the given elementary stresses act in the plane of greatest and least pressure. But in all actual problems respecting the stability of earth, the plane of greatest and least stress is known; and it is therefore unnecessary to apply to that subject the general problem as to the finding of the axes of pressure in space of three dimensions; a problem which requires the solution of a cubic equation.

§ 5. *Lemmata as to Conjugate Planes and Pressures.*

It is a well-known theorem in the theory of the elasticity of solids, that if the pressure on a given plane at a given point be parallel to a second plane, the pressure on the second plane at the same point must be parallel to the first plane. Such planes are said to be *conjugate* to each other, with respect to the pressures on them; the pressures also are said to be conjugate.

To adapt this theorem to the present question, the first step is to transform the equations (5.), so as to make the obliquity of the pressure, θ, the given angle, instead of the angle of direction ψ of the normal to the plane. Thus are obtained the following equations, from which, when the greatest and least pressures at a point are given, there may be found the position of a plane perpendicular to the plane of greatest and least pressure, on which the obliquity of the pressure shall be equal to a given angle θ; and also the amount R_u of the pressure corresponding to such obliquity.

$$\left.\begin{aligned} R_u &= M\cos\theta \mp \sqrt{D^2 - M^2\sin^2\theta} \\ 2\psi &= \frac{\pi}{2} + \theta \pm \cos^{-1}\frac{M\sin\theta}{D}. \end{aligned}\right\} \quad \ldots \ldots \quad (11.)$$

Hence it appears, that for each value of the obliquity θ, there are two values of ψ and two of R, the less value of ψ corresponding to the greater value of R, and conversely.

Let ψ_u be the *less*, and ψ_v the *greater* value of ψ, R_u the greater, and R_v the less value of R. Let the two normals Ou, Ov be drawn at opposite sides of the axis of greatest pressure Ox; then the angle between them is

$$uOv = \psi_u + \psi_v = \frac{\pi}{2} + \theta;$$

and the angle between the planes to which they are normal is . . . . . . (12.)

$$\pi - \psi_u - \psi_v = \frac{\pi}{2} - \theta;$$

therefore those two planes are *conjugate*.

Problem.—The positions of a pair of conjugate planes, both perpendicular to the plane of greatest and least pressure, and the pressures on them, being given, it is required to find the position of the axes of greatest and least pressure, and the magnitude of the greatest and least pressures.

From the equations (11.) it is easily deduced, that

$$\left.\begin{aligned} M &= \frac{R_u + R_v}{2\cos\theta} \\ D &= M\sqrt{\left\{\sin^2\theta + \left(\frac{R_u - R_v}{R_u + R_v}\cos\theta\right)^2\right\}} \\ &= \frac{1}{2}\cdot\sqrt{\{(R_u + R_v)^2\tan^2\theta + (R_u - R_v)^2\}} \\ 2\psi &= \frac{\pi}{2} + \theta \mp \cos^{-1}\frac{1}{\sqrt{\left\{1 + \left(\frac{R_u - R_v}{R_u + R_v}\cot\theta\right)^2\right\}}}. \end{aligned}\right\} \quad \ldots \ldots \quad (13.)$$

The axis of greatest pressure will be found in the obtuse angle between the normals Ou, Ov, and nearer to Ou, the normal to the plane on which the pressure is the greater.

When $R_u = R_v$, then $\theta = \Theta$, the angle of greatest obliquity. In this case let

$$R_u = R_v = R(\Psi),$$

of which the value, in terms of M and D, has already been given. Then it appears that

$$\left.\begin{aligned} M &= \frac{R(\Psi)}{\cos\Theta} \\ D &= M\sin\Theta = R(\Psi)\tan\Theta \\ 2\Psi &= \frac{\pi}{2}+\Theta, \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (14.)$$

in accordance with the equations (7.) and (8.).

From the equations (13.) and (14.) it is easily deduced that the ratio of a pair of conjugate pressures has the following value:—

$$\frac{R_v}{R_u} = \frac{\cos\theta - \sqrt{\sin^2\Theta - \sin^2\theta}}{\cos\theta + \sqrt{\sin^2\Theta - \sin^2\theta}}. \quad \ldots \ldots \ldots \quad (15.)$$

§ 6. *Lemmata as to the Internal Equilibrium of a Solid Mass.*

Let Ox', Oy', Oz' be rectangular axes, of which Ox' is vertical, and positive downwards; and let G be the weight of unity of volume of a solid mass. Then the well-known conditions of the internal equilibrium of such a mass are the following:—

$$\left.\begin{aligned} \frac{dP_{x'}}{dx'}+\frac{dQ_{z'}}{dy'}+\frac{dQ_{y'}}{dz'} &= G; \\ \frac{dQ_{z'}}{dx'}+\frac{dP_{y'}}{dy'}+\frac{dQ_{x'}}{dz'} &= 0; \\ \frac{dQ_{y'}}{dx'}+\frac{dQ_{x'}}{dy'}+\frac{dP_{z'}}{dz'} &= 0. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (16.)$$

In all actual problems respecting the stability of earth, the plane of greatest and least pressures is vertical, and there is one horizontal direction, normal to that plane, along which the state of stress of the earth does not vary. It will be sufficient, therefore, to restrict the above equations to two dimensions, by making $Q_{x'}=0$; $Q_{y'}=0$; $\frac{d}{dz'}=0$; and putting Q simply for $Q_{z'}$. Then we have the two equations,—

$$\left.\begin{aligned} \frac{dP_{x'}}{dx'}+\frac{dQ}{dy'} &= G; \\ \frac{dQ}{dx'}+\frac{dP_{y'}}{dy'} &= 0. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (17.)$$

§ 7. *Surfaces of Uniform Horizontal Thrust.*

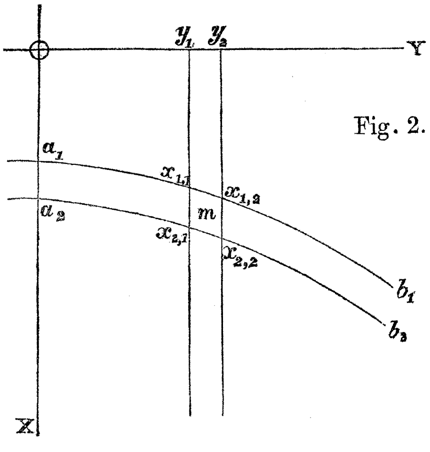


Fig. 2.

The following is a peculiar transformation of these differential equations, suited to the subject of the present investigation. OX, fig. 2, being vertical, and OY horizontal, and in the plane of greatest and least pressure, conceive the mass to be subdivided into prismatic molecules by an indefinite number of vertical planes perpendicular to the plane XY, such as $y_1x_{1\,1}x_{2,\,1}$, $y_2x_{1\,2}x_{2,\,2}$, and by an indefinite number of surfaces, also perpendicular to the plane XY, such as a_1b_1, a_2b_2, and of such a figure, that the tangent plane at each point of each of those surfaces is conjugate to a vertical plane, in the sense explained in section 5. Let

$$a = Oa$$

be the vertical ordinate of any one of those surfaces corresponding to $y=0$; then

$$x = f(a, y) \quad \ldots \ldots \quad (18.)$$

will be its equation. By the definition of conjugate planes, the pressure on each element of such a surface is vertical. Let R_u be its amount per unit of area of the surface, and R_v the pressure on unity of area of a vertical plane, which pressure is parallel to a tangent to ab; the angle of obliquity being given by the equation

$$\tan\theta = \frac{dx}{dy}. \quad \ldots \ldots \quad (19.)$$

Let X be the vertical pressure on a given element of a surface ab, per unit of area of the *projection of that surface on a horizontal plane*; then

$$X = \frac{R_u}{\cos\theta} = R_u \sqrt{1+\frac{dx^2}{dy^2}}. \quad \ldots \ldots \quad (20.)$$

It is evident that the equations of the equilibrium of a prismatic element m are the following:—

$$\left.\begin{aligned} \frac{d}{da}(X - Gx) + \frac{d}{dy}\left(R_v . \sin\theta \cdot \frac{dx}{da}\right) = 0; \\ \frac{d}{dy}\left(R_v \cos\theta \frac{dx}{da}\right) = 0. \end{aligned}\right\} \quad \ldots \ldots \quad (21.)$$

The second of these equations being integrated, gives

$$R_v \cos\theta \frac{dx}{da} = \frac{R_v \dfrac{dx}{da}}{\sqrt{1+\dfrac{dx^2}{dy^2}}} = F(a), \quad \ldots \ldots \quad (22.)$$

which value being introduced into the first equation, gives the following,

$$\frac{d}{da}(X - Gx) + F(a) \cdot \frac{d^2x}{dy^2} = 0. \quad \ldots \ldots \quad (23.)$$

Now let

$$H = \int_0^a F(a)da = \int_0^a R_v \cos\theta \frac{dx}{da} da \quad \ldots \ldots \quad (24.)$$

be the *total horizontal thrust* of the solid mass from its upper surface, down to the surface under consideration. This quantity, being independent of y, may be used as an independent variable instead of a; that is to say, dividing equation (23.) by $F(a)$, we obtain the following:—

$$\frac{d}{dH}(Gx - X) = \frac{d^2x}{dy^2}, \quad \ldots \ldots \quad (25.)$$

which is the differential equation of a *Surface of Uniform Thrust.*

§ 8. *Surfaces of Uniform Thrust and Uniform Vertical Pressure.*

With an exception to be described in the next section, the only case in which the equation (25.) becomes linear with respect to x, and capable of being satisfied by an indefinite number of arbitrary forms of surface, is that in which each surface of uniform thrust is also a surface of uniform vertical pressure; that is to say, when

$$X=F(H). \quad \quad (26.)$$

In this case, the integral of equation (25.), as found by the method of FOURIER, is capable of being expressed in various forms, of which the following is the most comprehensive:—

$$x=\frac{F(H)}{G}+\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-m^2}f\left(y+2m\sqrt{\frac{H}{G}}\right)dm; \quad \quad (27.)$$

the function f being such, that neither

$$e^{-m^2}f, \text{ nor } e^{-m^2}f',$$

shall become infinite for any value of the argument, how great soever, nor increase indefinitely with the argument, and that they shall both vanish at the limits of integration. This function is determined by the following condition at the upper surface of the mass:—

$$x_0=f(y). \quad \quad (28.)$$

In all those cases in which the upper surface of the mass deviates alternately above and below an inclined or horizontal plane by deviations which recur periodically in each horizontal distance 2B, the integral of the differential equation (25.) is capable of being expressed in the following form:—

$$x=\frac{F(H)}{G}+Ay+\Sigma . e^{-\frac{n^2\pi^2H}{GB^2}}\left(C_n\sin\frac{n\pi y}{B}+C'_n\cos\frac{n\pi y}{B}\right), \quad \quad (29.)$$

where A is the tangent of the average declivity, above and below which the surfaces of equal thrust deviate periodically, and C_n and C'_n are determined by the following formulæ:—

$$\left.\begin{array}{l} C_n=\frac{2}{B}\int_{-\frac{B}{2}}^{\frac{B}{2}}(x_0-Ay)\sin\frac{n\pi y}{B}.dy \\ C'_n=\frac{2}{B}\int_{-\frac{B}{2}}^{\frac{B}{2}}(x_0-Ay)\cos\frac{n\pi y}{B}.dy, \end{array}\right\} \quad \quad (30.)$$

Such are the integrals of the equations of internal equilibrium in two dimensions in a vertical plane, in those cases in which each surface of uniform thrust is also a surface of uniform vertical pressure; a condition realized in those cases in which the horizontal thrust is caused by the vertical pressure.

The relation $X=F(H)$ between the horizontal thrust and the vertical pressure, still remains to be determined by the physical conditions of each particular problem.

NOTE. *Equilibrium of Arched Ribs.*

Although foreign to the immediate subject, the fact may here be noted, that by making $X=F(H)=0$ in equations (27.) and (29.), they furnish the means of determining the form of the intrados of any equilibrated arched rib loaded solely with its own weight, or proportionally to the weight of each of its parts, when the form of the extrados is either represented by equation (28.), or developed by means of the formulæ (30.); care being taken, in the latter case, to include the entire rib within the limits $y=\pm\frac{1}{2}B$. If the rib be divided into voussoirs, their joints ought theoretically to be normal to the series of curves given by equation (27.) or equation (29.).

§ 9. *Extrinsic and Intrinsic Vertical Pressures.*

The exception referred to at the beginning of the last section is as follows. Besides the vertical pressure

$$X_H=F(H),$$

which is uniform for each surface of equal thrust, there may also be a vertical pressure, uniform in each vertical line, and varying proportionally to an arbitrary function of the horizontal coordinate, viz.—

$$X_y=\Phi(y). \quad \ldots \ldots \ldots \ldots \quad (31.)$$

This last pressure does not enter into the differential equation (25.), and its amount is therefore to be determined in each problem by conditions distinct from the general conditions of internal equilibrium. It may be called the *Extrinsic Vertical Pressure*; and the vertical pressure depending on the horizontal thrust, the *Intrinsic Vertical Pressure*; so that the surfaces determined in the last section are surfaces of *uniform intrinsic vertical pressure*, the extrinsic vertical pressure being independent of their form and position*.

§ 10. *Limiting Relations of the Pressures, at a point in a Mass of loose Earth.*

The seven preceding sections, from § 3 to § 9 inclusive, contain principles which are common to every case of the internal equilibrium of a solid mass, in what manner soever constituted.

In the present section, those common principles will be combined with the peculiar principle of the stability of an incoherent granular mass, explained in §§ 1 and 2.

The symbol $\lesseqgtr$ will be used in the following sense:—

$$a \lesseqgtr b \pm c$$

means, "a is *not less* than $b-c$, and *not greater* than $b+c$." The symbols $\leq$ and $\geq$ will be used in their ordinary sense of "not greater than" and "not less than."

φ being, as in § 2, the angle of repose of a given kind of earth, and Θ, as in § 3, the

* In the sequel, when vertical pressures are not expressly designated as extrinsic, they are to be understood as intrinsic.

greatest obliquity of the pressure at any point in any plane traversing that mass, it appears from equation (2.) that the condition of stability of the mass is

$$\Theta \leq \varphi. \qquad (32.)$$

From this condition the following propositions are deduced:—

THEOREM I. *At each point in a mass of earth, the ratio of the difference of the greatest and least pressures to their sum cannot exceed the sine of the angle of repose.*

This theorem* follows from the second of the equations (7.), its symbolical expression being

$$\frac{P_x - P_y}{P_x + P_y} = \frac{D}{M} \leq \sin \varphi. \qquad (33.)$$

THEOREM II. *The following is the expression of the condition of stability of a mass of earth, in terms of the pressures at a point, referred to any pair of rectangular axes,* Ox′, Oy′, *in the plane of greatest and least pressures:—*

$$\frac{(P_{x'} - P_{y'})^2 + 4Q'^2}{(P_{x'} + P_{y'})^2} = \frac{D^2}{M^2} \leq \sin^2 \varphi. \qquad (34.)$$

This follows from the equations (10.).

THEOREM III. *The following is the expression of the condition of the stability of a mass of earth, in terms of the ratio of a pair of conjugate pressures in the plane of greatest and least pressures:—*

Let R_u, R_v be the two conjugate pressures, θ their common angle of obliquity; then

$$\frac{R_v}{R_u} \gtrless \frac{\cos\theta \mp \sqrt{\cos^2\theta - \cos^2\varphi}}{\cos\theta \pm \sqrt{\cos^2\theta - \cos^2\varphi}}. \qquad (35.)$$

This follows from equation (15.).

THEOREM IV. *The positions of a pair of conjugate planes being given, the following are the limits, consistent with stability, of the angle which the axis of greatest pressure can make with either of the normals to the conjugate planes:—*

$$\psi \lesseqgtr \frac{\pi}{4} + \frac{\theta}{2} \pm \frac{1}{2} \cos^{-1} \cdot \frac{\sin\theta}{\sin\varphi}. \qquad (36.)$$

This follows from the second of the equations (11.).

THEOREM V. *The amount* R *and obliquity* θ *of the pressure on a given plane being given, the following are the limits, consistent with stability, of the half sum* M *and half difference* D, *of the greatest and least principal pressures:—*

$$\left.\begin{array}{l} M \lessgtr \dfrac{R}{\cos\theta \mp \sqrt{\cos^2\theta - \cos^2\varphi}} \\ D = M\sin\theta \lessgtr \dfrac{R\sin\varphi}{\cos\theta \mp \sqrt{\cos^2\theta - \cos^2\varphi}} \end{array}\right\} \qquad (37.)$$

This follows from the first of the equations (11.).

The greater values of M and D given by this equation correspond to the greater values of the angle ψ given by equation (36.).

* Already published in the 'Proceedings of the Royal Society' for the 6th of March, 1856.

Corollary to the Theorems III., IV., and V.—When the angle of obliquity θ is equal to the angle of repose φ, the quantities given by the equations (35.), (36.), (37.), have each but one value, without any limits of deviation, viz.—

$$\left.\begin{array}{l}\frac{R_v}{R_u}=1;\ \Psi=\frac{\pi}{4}+\frac{\varphi}{2};\\ M=R\sec\varphi;\ D=R\tan\varphi.\end{array}\right\} \quad (38.)$$

§ 11. *Planes of Rupture.*

The angle Ψ, above given, is that made on either side of the axis of greatest pressure by the normals to the pair of planes along which the tendency of the earth to give way by sliding is greatest. The angle made by these planes themselves on either side of the axis of greatest pressure is therefore

$$\Psi-\varphi=\frac{\pi}{4}-\frac{\varphi}{2}. \quad (39.)$$

Those planes are called *Planes of Rupture.* Their position, in the particular case of a horizontal bank, where the axis of greatest pressure is vertical, was determined by COULOMB by the aid of his ideal *wedge of least resistance.*

§ 12. *Application of* MR. MOSELEY'S *Principle of Least Resistance to the Stability of Earth.*

This principle may be stated as follows:—The forces which balance each other in or upon a given body or structure being distinguished into two systems, called respectively *active* and *passive*, which stand to each other in the relation of cause and effect, then will the passive forces be the least which are capable of balancing the active forces, consistently with the physical condition of the body or structure.

In a mass of earth, the active forces are the vertical pressures produced by the gravitation of its parts; the passive forces are the pressures conjugate to those vertical pressures, whereby the earth is prevented from spreading. The pressures conjugate to the vertical pressures will therefore be the least which are at once consistent with the conditions of internal equilibrium given in §§ 6, 7, 8, and 9, and with the conditions of stability at each point, given in § 10.

§ 13. *Statement of the General Problem of the Stability of a Mass of Earth under its own Weight.*

The upper and free surface of the mass of earth, at which the intrinsic vertical pressure $X_H=0$, is supposed to be curved in one coordinate plane only, that of greatest and least pressures; and the form of the section of that surface by the vertical plane of greatest and least pressures is supposed to be given by an equation of the form (28.).

The specific gravity G and angle of repose φ of the earth being given, it is required to determine the form and position of the surface at which the intrinsic vertical pressure has any given value.

In order that the equations (27.), or (29.) and (30.) may furnish the complete solution of this problem, it is necessary now to determine, from the conditions of stability in § 10, the relation $X_H = F(H)$, between the intrinsic vertical pressure and the horizontal thrust.

The case in which the upper surface of the earth is an indefinitely extended plane, horizontal or inclined, is the only case which admits of an exact solution. It will therefore be solved first, and its solution used to facilitate the solution of the more complex case, which is solved approximately by the integral (27.), or by infinite series of the form (29.).

§ 14. Problem I. *Surfaces of Equal Pressure and Thrust in the case of a horizontal or uniformly sloping bank.*

In this case, equation (28.), giving the form of the free surface, becomes

$$x_0 = Ay = y \tan \theta. \qquad (40.)$$

Equation (27.) gives for the form and position of any surface of uniform intrinsic vertical pressure,

$$x = \frac{X_H}{G} + Ay; \text{ or } x - x_0 = \frac{X_H}{G}. \qquad (41.)$$

Hence the surfaces of equal intrinsic vertical pressure are planes parallel to the free surface, and the vertical pressure is simply the weight of a column of earth of unity of area of base, and of the height $x - x_0$. At each point, a vertical plane, and a plane parallel to the free surface, are conjugate to each other; that is, the pressure on a plane parallel to the surface is vertical, and the pressure on a vertical plane is parallel to the surface; and the angle of slope, θ, is the common angle of obliquity of those conjugate pressures.

Equation (20.) gives for the vertical pressure per unit of area of a plane parallel to the surface,

$$R_u = X \cos \theta = G(x - x_0) \cos \theta. \qquad (42.)$$

Now from the principle of least resistance, it follows that the pressure at any point on a vertical plane, in a direction parallel to the slope, must have the least value consistent with the equation of stability (35.); that is to say,

$$R_v = R_u \cdot \frac{\cos \theta - \sqrt{\cos^2 \theta - \cos^2 \varphi}}{\cos \theta + \sqrt{\cos^2 \theta - \cos^2 \varphi}}. \qquad (43.)$$

For brevity's sake, let

$$\cos^2 \theta \cdot \frac{\cos \theta - \sqrt{\cos^2 \theta - \cos^2 \varphi}}{\cos \theta + \sqrt{\cos^2 \theta - \cos^2 \varphi}} = k. \qquad (44.)$$

Then the horizontal component of R_v has the following value:—

$$\frac{dH}{dx} = R_v \cos \theta = kX = kG(x - x_0). \qquad (45.)$$

And the total horizontal thrust, from the surface down to a given depth $x-x_0$, is

$$H=\frac{kG(x-x_0)^2}{2} \quad \quad (46.)$$

The relation, then, between the vertical pressure and the total horizontal thrust down to a given surface of uniform thrust and pressure, is expressed by the equation

$$H=\frac{kX^2}{2G}, \text{ or } X=\sqrt{\frac{2GH}{k}}. \quad \quad (47.)$$

Equation (36.) gives, for the angle made by the axis of greatest pressure at each point with the vertical,

$$\frac{\pi}{2}-\psi=\frac{\pi}{4}-\frac{\theta}{2}-\frac{1}{2}\cos^{-1}\cdot\frac{\sin\theta}{\sin\varphi}. \quad \quad (48.)$$

This axis lies in the acute angle between the slope of the surface and the vertical.

The two planes of rupture at each point make with this axis the angle given by equation (39.); that is to say, they make with the vertical the following angles respectively at opposite sides:—

$$\left.\begin{array}{l}\frac{\pi}{2}-\frac{\theta}{2}-\frac{1}{2}\cos^{-1}\frac{\sin\theta}{\sin\varphi}-\frac{\varphi}{2}, \text{ and} \\ \frac{\theta}{2}+\frac{1}{2}\cos^{-1}\frac{\sin\theta}{\sin\varphi}-\frac{\varphi}{2}.\end{array}\right\} \quad \quad (49.)$$

§ 15. *Extreme cases of this Problem.*

The two extreme cases of this problem are respectively, when the free surface is horizontal, or $\theta=0$, and when it slopes at the angle of repose, or $\theta=\varphi$.

The following are the results in these two cases.

Case First. $\theta=0$:

$$\left.\begin{array}{l}R_u=X;\ R_v=X\frac{1-\sin\varphi}{1+\sin\varphi};\ k=\frac{1-\sin\varphi}{1+\sin\varphi}; \\ \frac{\pi}{2}-\psi=0, \text{ or the axis of greatest pressure is vertical, and the planes} \\ \text{of rupture make the angles } \frac{\pi}{4}-\frac{\varphi}{2} \text{ on either side of the vertical.}\end{array}\right\} \quad \quad (50.)$$

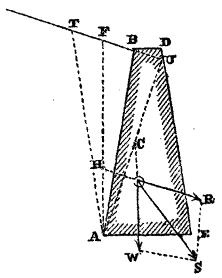
Case Second. $\theta=\varphi$:

$$\left.\begin{array}{l}R_u=X\cos\varphi;\ R_v=R_u=X\cos\varphi;\ k=\cos^2\varphi; \\ \frac{\pi}{2}-\psi=\frac{\pi}{4}-\frac{\varphi}{2}, \text{ or the axis of greatest pressure bisects the acute angle} \\ \text{between the vertical and the slope of the surface, and the planes of} \\ \text{rupture are respectively vertical, and parallel to the surface.}\end{array}\right\} \quad \quad (51.)$$

§ 16. *Revêtement Walls.* PROBLEM II. *To determine the magnitude and direction of the resultant pressure on the base of the revêtement wall of an earthen bank, having a horizontal or uniformly sloping upper surface, the angle of repose of the earth upon the masonry being not less than that of the earth upon itself.*

Case 1. When the back of the wall does *not* overhang the base. (See fig. 3.)

Fig. 3.



Let ABDE be the section of the wall. Through A draw AF vertical ($=x-x_0$), cutting the upper surface of the bank in F. If the back of the wall is vertical, F coincides with B.

Through C, the centre of gravity of the mass of masonry and earth AFBDE, draw CW vertical. Take $\overline{AH}=\frac{1}{3}\overline{AF}$, and draw HR parallel to the surface of the bank, cutting CW in O; this will be the position and direction of the resultant pressure on the plane AF. To represent that resultant pressure, take

$$\overline{OR}=\frac{H}{\cos\theta}=\frac{kG.\overline{AF}^2}{2\cos\theta} \quad \text{. . . . . (52.)}$$

Also take $\overline{OW}$ to represent the weight of the mass of earth and masonry AFBDE; then will the resultant $\overline{OS}$ of $\overline{OR}$ and $\overline{OW}$ represent in magnitude and direction the resultant pressure required on the base AE.

$$\left.\begin{aligned}\overline{OS}&=\sqrt{\left\{\overline{OR}^2+\overline{OW}^2+2\overline{OR}.\overline{OW}.\sin\theta\right\}};\\ \sin\angle WOS&=\frac{H}{\overline{OS}}:\end{aligned}\right\} \quad \text{. . . . . (53.)}$$

$\sin\theta$ is to be taken as positive when the bank slopes down towards the wall.

Case 2. When the back of the wall overhangs, as represented, for example, by AT.

Proceed in all respects as above, except that in finding the centre of gravity C, and the weight $\overline{OW}$, the mass AFBDE alone is to be treated as masonry, and the prism AFT is to be treated as if its specific gravity were merely the *excess* of the specific gravity of the masonry above that of the earth. This is because a pressure equal to the weight of the earth which the prism AFT could contain, is sustained by the earth vertically below it, leaving only the excess of the weight of masonry over that of earth to add to the stability of the wall. As this excess is in general very small, it follows that there is in general little or no advantage in building revêtement walls so as to overhang behind. For the same reason it is evident, that if a straight line be drawn from A to U, where the line of slope cuts the face of the wall, the masonry behind this line contributes little to the stability of the wall.

§ 17. *General Case:—its ambiguity.*

The application of the principle of least resistance to the case in which the vertical section of the upper surface of the earth is of any form (limited only by the condition that the slope shall nowhere exceed the angle of repose), is attended with this difficulty;—that at certain portions of each layer of equal thrust, viz. the lower portion, it is difficult, if not impossible, to determine which of the two conjugate pressures, the vertical pressure and the horizontal or inclined pressure, is to be regarded as *cause* and which as *effect*; and thus the solution becomes ambiguous.

There is one case, however, which is not affected by such ambiguity; viz. that in which the steepest declivity of each surface of equal thrust is the angle of repose φ; for in that case there is but one relation, consistent with stability, between the vertical pressure and horizontal thrust, without any limits of deviation.

The relation between those forces which is *the only solution* in the case specified, is *one of the solutions* consistent with stability in every other case, as will appear from the following

LEMMA. *The intensity of the horizontal thrust*, $R_v \cos\theta = X\cos^2\varphi$, *which corresponds with a given vertical pressure*, X, *for a declivity sloping at the angle of repose*, φ, *lies between the limits consistent with stability for every declivity sloping at a less angle.*

For the condition of stability deduced from equation (35.) is

$$\frac{R_v\cos\theta}{X}=\frac{R_v}{R_u}\cos^2\theta \gtrless \cos^2\theta\cdot\frac{1\mp\sqrt{1-\frac{\cos^2\varphi}{\cos^2\theta}}}{1\pm\sqrt{1-\frac{\cos^2\varphi}{\cos^2\theta}}} \gtrless \frac{\cos^2\varphi}{\left(1\pm\sqrt{1-\frac{\cos^2\varphi}{\cos^2\theta}}\right)^2}; \quad . \quad . \quad . \quad . \quad (54.)$$

and $\cos^2\varphi$ is always within the limits fixed by this equation.

§ 18. PROBLEM III. *The vertical section of the upper surface of a mass of earth, curved in the plane of section only, being given, it is required to find the form and position of the surfaces of uniform vertical pressure, when the greatest declivity of each of them is the angle of repose.*

It appears from the preceding lemma that the solution of this problem is always *one of the solutions* even when the greatest declivity is less than the angle of repose.

Let the equation of the upper surface be expressed by the equation (28.), or developed by the formulæ (30.). Then the *form* of the surfaces of equal pressure is given by the transcendental parts of the equations (27.) or (29.); and it only remains to determine their *position*, by finding the relation between the vertical pressure X and the horizontal thrust H.

The condition of stability gives

$$X=\frac{R_v\cos\theta}{\cos^2\varphi}=\frac{1}{\cos^2\varphi\,\frac{dx}{dH}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55.)$$

From the general differential equation (25.), it appears that

$$\frac{dx}{dH}=\frac{1}{G}\left(\frac{dX}{dH}+\frac{d^2x}{dy^2}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56.)$$

but at the points of greatest declivity of each surface

$$\frac{d^2x}{dy^2}=0,$$

consequently equation (55.) becomes

$$X=\frac{G}{\cos^2\varphi\frac{dX}{dH}},$$

which being integrated, gives

$$H=\frac{\cos^2\varphi\,.\,X^2}{2G}; \quad X=\frac{\sqrt{2GH}}{\cos\varphi}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57.)$$

being the solution required.

The introduction of the above value of the thrust into the equations (27.) and (29.) gives the following results, showing at once the relative position and the form of the surfaces of equal pressure:—

$$\left.\begin{array}{l}\text{General}\\ \text{Solution}\end{array}\right\}x=\frac{\mathrm{X}}{\mathrm{G}}+\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-m^2}f\left(y+\frac{\sqrt{2}.m\mathrm{X}\cos\phi}{\mathrm{G}}\right)dm. \quad \ldots\ldots \quad (58.)$$

$$\left.\begin{array}{l}\text{Periodical}\\ \text{Solution}\end{array}\right\}x=\frac{\mathrm{X}}{\mathrm{G}}+\mathrm{A}y+\Sigma\cdot e^{-\frac{1}{2}\left(\frac{n\pi\mathrm{X}\cos\phi}{\mathrm{GB}}\right)^2}\left(\mathrm{C}_n\sin\frac{n\pi y}{\mathrm{B}}+\mathrm{C}'_n\cos\frac{n\pi y}{\mathrm{B}}\right). \quad \ldots \quad (59.)$$

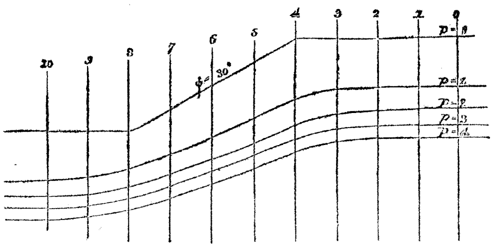
In using the first of the above equations, it is to be observed that

$$\int_{-\infty}^{+\infty}e^{-m^2}dm=\sqrt{\pi}. \quad \ldots\ldots\ldots\ldots \quad (60.)$$

§ 19. *Approximate Description of Surfaces of Equal Pressure by Graphic Construction.*

The following method of approximately determining the forms of the surfaces of equal pressure or equal thrust, is analogous to that employed by Professor WILLIAM THOMSON in some recent researches on Electricity, for the purpose of finding the successive forms of the curve showing the electric tension in a conductor.

Fig. 4.



Let $x=\frac{\mathrm{X}}{\mathrm{G}}+x_y$. Then, by the general equations (25.), (26.), we have

$$\mathrm{G}\frac{dx_y}{d\mathrm{H}}=\frac{d^2x_y}{dy^2}.$$

Substituting finite differences for differentials, it follows that the subjoined equation is approximately true:—

$$\frac{\Delta_{\mathrm{H}}x_y}{\Delta_y^2x_y}=\frac{\Delta\mathrm{H}}{\mathrm{G}\Delta y^2}; \quad \ldots\ldots\ldots\ldots \quad (61.)$$

Δ_{H} denoting differences caused by varying H, and Δ_y those caused by varying y.

Now let the vertical section of any surface of equal thrust be given; and let it be intersected by equidistant vertical ordinates, their uniform interval in a horizontal direction being Δy. Join by a straight chord any two of the points of intersection whose distance apart is two intervals, or $2\Delta y$. The vertical sagitta intercepted upon the intermediate ordinate between the chord and the given line of section will be $\frac{1}{2}\Delta_y^2x_y$; and if a second surface of equal thrust be conceived to pass through the middle of the chord, the same sagitta will be $\Delta_{\mathrm{H}}x_y$. Consequently, for the pair of surfaces which respectively traverse the ends and the middle of the chord in question,

$$\frac{\Delta_{\mathrm{H}}x_y}{\Delta_y^2x_y}=\frac{\Delta\mathrm{H}}{\mathrm{G}\Delta y^2}=\frac{1}{2} \quad \ldots\ldots\ldots\ldots \quad (62.)$$

Hence we have the following construction. Inscribe a polygon in the vertical section of the first surface of equal thrust by joining the points where the surface is intersected by alternate ordinates: the points where the intermediate ordinates bisect the sides of the polygon will lie very nearly in a curve parallel to a surface of equal thrust at which the total thrust is greater than at the first surface by the amount $\frac{1}{2}G\Delta y^2$. Inscribe a polygon in the second curve by joining the points found in it; the points of bisection of the sides of this second polygon will indicate a third curve, and so on; the constant increment of the thrust from any curve to the next being still $\frac{1}{2}G\Delta y^2$.

Thus are drawn approximately the *forms* of a series of surfaces of equal thrust. Their *distances apart* are to be computed by equation (57.), as before.

Let the first surface be the upper surface of the mass of earth, where $H=0$; then the successive inscription of the number of polygons denoted by p will show the form of the surface at which the total horizontal thrust is

$$H=\frac{p}{2}G\Delta y^2; \quad \ldots \ldots \ldots \ldots \quad (63.)$$

and the mean depth of this surface below the upper surface (that is to say, the depth to which the form of each surface is to be transferred vertically downwards, after having been drawn by means of the inscription of polygons,) will be given by the formula

$$\frac{X}{G}=\sec\varphi\sqrt{\frac{2H}{G}}=\sqrt{p}.\sec\varphi.\Delta y. \quad \ldots \ldots \ldots \quad (64.)$$

Fig. 4 shows the result of applying this process to an outline of upper surface marked $p=0$, consisting partly of two horizontal planes, and partly of a declivity sloping at an angle of 30°, which is also supposed to be the angle of repose.

§ 20. *Of the Limits of Extrinsic Pressure and the Stability of Foundations in Earth.*

At any point on a surface of uniform thrust and uniform vertical pressure, where the declivity θ is less than the angle of repose φ, it appears from the equations (35.) and (54.) that there are two limits to the ratio of the vertical to the horizontal pressure, given by the equation

$$X\frac{dx}{dH}\substack{<\\>}\frac{1}{\cos^2\theta}\cdot\frac{1\pm\sqrt{1-\frac{\cos^2\varphi}{\cos^2\theta}}}{1\mp\sqrt{1-\frac{\cos^2\varphi}{\cos^2\theta}}}$$

$$\substack{<\\>}\frac{1}{\cos^2\varphi}\cdot\left(1\pm\sqrt{1-\frac{\cos^2\varphi}{\cos^2\theta}}\right)^2. \quad \ldots \ldots \ldots \quad (65.)$$

Should, therefore, the intrinsic vertical pressure X_H due to the weight of the earth be less than that fixed by the upper limit of the above expression, an *extrinsic vertical pressure* may be applied at the given point of the given layer (by means, for example, of the foundation of a building) not exceeding the amount whereby the intrinsic pressure falls short of the limit; that is to say,—

$$X_y\leq\frac{1}{\frac{dx}{dH}\cos^2\varphi}\cdot\left(1+\sqrt{1-\frac{\cos^2\varphi}{\cos^2\theta}}\right)^2-X_H. \quad \ldots \ldots \ldots \quad (66.)$$

In the case to which Problem III. refers, in which

$$X_H = \frac{1}{\frac{dx}{dH}\cos^2\varphi},$$

equation (66.) becomes

$$X_y \leq \frac{1}{\frac{dx}{dH}\cos^2\varphi} \cdot \left\{1 - \frac{\cos^2\varphi}{\cos^2\theta} + 2\sqrt{1 - \frac{\cos^2\varphi}{\cos^2\theta}}\right\}. \qquad (67.)$$

The most important application of these principles relates to the power of a mass of earth whose upper surface is horizontal, to sustain the weight of a building founded at a given depth.

In this case, the condition of stability expressed by the equation (65.) becomes

$$X\frac{dx}{dH} \lessgtr \frac{1 \pm \sin\varphi}{1 \mp \sin\varphi}. \qquad (68.)$$

Now the horizontal pressure of the earth may be so increased by ramming, as to have the maximum amount consistent with the intrinsic vertical pressure due to the weight of the earth, in which case

$$X_H \frac{dx}{dH} = \frac{1 - \sin\varphi}{1 + \sin\varphi}. \qquad (69.)$$

And consequently, the extrinsic vertical pressure, due to the excess of the weight of a building above that of the earth which it displaces, is limited by the equation

$$X_y \leq \frac{4\tan\varphi}{\frac{dx}{dH}\cos\varphi} \leq X_H \cdot \frac{4\sin\varphi}{(1-\sin\varphi)^2}; \qquad (70.)$$

and the *ratio of the weight of a building, to the weight of the earth displaced by its foundation*, is limited by the equation

$$\frac{X_H + X_y}{X_H} \leq \left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)^2. \qquad (71.)$$

When the angle of repose, for example, is 30°, the limit of that ratio is 9.

§ 21. *Negative Extrinsic Pressure. Resistance of Screw Piles to Extraction.*

When the intrinsic vertical pressure exceeds the lower limit given by the equation (65.), the earth will resist a *Negative Extrinsic Pressure*, or upward tension, not exceeding such excess. To avoid the use of negative signs, let

$$-X_y = T_y;$$

then

$$T_y \leq X_H - \frac{1}{\frac{dx}{dH}\cos^2\varphi} \cdot \left(1 - \sqrt{1 - \frac{\cos^2\varphi}{\cos^2\theta}}\right)^2. \qquad (72.)$$

Let the surface of the earth be horizontal, and let the horizontal pressure have the least amount consistent with stability. Then

$$X_H \frac{dx}{dH} = \frac{1+\sin\varphi}{1-\sin\varphi}, \quad \ldots \ldots \ldots \quad (73.)$$

and

$$T_y \leq \frac{4\tan\varphi}{\frac{dx}{dH}\cos\varphi} \leq X_H \cdot \frac{4\sin\varphi}{(1+\sin\varphi)^2}. \quad \ldots \ldots \ldots \quad (74.)$$

The last expression shows the ratio borne to the superincumbent load, by the upward tension necessary to extract from the earth a body with a nearly horizontal upper surface, like the thread of a screw pile. When the angle of repose, for example, is 30°, that ratio is $\frac{8}{9}$.

§ 22. *Resistance of Earth to the Horizontal Thrust of a Building.*

The limit of the resistance of a horizontal stratum of earth to the horizontal thrust of the foundation of a building, is the upper limit of the horizontal thrust of the stratum, consistent with its weight; that is to say,—

$$H \leq \frac{X^2}{2G} \cdot \frac{1+\sin\varphi}{1-\sin\varphi} = \frac{G(x-x_0)^2}{2} \cdot \frac{1+\sin\varphi}{1-\sin\varphi}. \quad \ldots \ldots \ldots \quad (75.)$$